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ENERGY RELEASE RATE FOR CRACKS IN IDEAL COMPOSITES

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Abstract—Infinitesimal plane deformations of ideal fiber-reinforced composites with elastic shearing stress response are considered. The fibers are straight and parallel, and there is a straight crack perpendicular to the fibers. A general expression for the energy release rate per unit length of crack advance is obtained. Explicit expressions in terms of the body geometry and loading are obtained for three special classes of body shapes: bodies symmetrical about a fiber, bodies bounded on the cracked side by a fiber, and bodies bounded on the opposite side by a fiber. The results also apply to cracks parallel to the fibers, and to cracks in compressible materials reinforced by two orthogonal families of inextensible fibers.

1. INTRODUCTION

Materials composed of strong fibers bonded together with a weaker material can be modelled by treating the fibers as continuously distributed and inextensible. The analysis of deformations of such materials is greatly simplified if it is also assumed that the composite is incompressible in bulk. An idealized material satisfying these assumptions is called an ideal fiber-reinforced composite. The theory of such materials is the subject of a book by Spencer[1] and a recent review article[2]. The applicability of the theory to real composites is discussed in many of the papers that we cite.

The theory is especially simple when the fibers are straight and parallel, the shearing stress response of the material is elastic, and attention is restricted to infinitesimal plane deformations. We use these assumptions in the present paper. In such cases displacement boundary value problems are trivial, and explicit solutions of many traction boundary value problems have been obtained[3–5]. It is known that the plane traction problem is well-set[6].

The work of England and Rogers[4] concerns cracked bodies, and contains complete solutions for bodies with two orthogonal axes of reflectional symmetry, the cracks being either parallel or perpendicular to the fiber direction. Thomas and England[5] have worked out a case involving a crack oblique to the fibers.

In the present paper we extend this work on plane crack problems by deriving a simple general expression for the energy release rate in crack advance, analogous to the expression for the energy release rate in terms of the stress intensity factor used in elastic fracture mechanics[7, 8]. We then derive explicit expressions in terms of the body geometry and loading, for some special classes of body shapes.

We consider the plane deformation of a body of arbitrary shape with a straight crack perpendicular to the fiber direction (Section 2). Because of the well-known duality between fibers and their orthogonal trajectories in plane deformations, this problem is mathematically equivalent to one involving a crack parallel to the fibers, and to plane stress or plane strain in compressible materials reinforced by two orthogonal families of inextensible fibers. We choose to discuss the least likely application of the theory because the language is simplest in that case.

In Section 3 we show that the energy released per unit length of crack advance is equal to F_0^2/GL^*, where F_0 is the force at the crack tip in the fiber passing through it, G is the shear modulus of the material, and L^* is the harmonic mean length of the two parts of the crack-tip fiber.

In the remainder of the paper we derive explicit expressions for F_0 for some special classes of body shapes. We show that for bodies symmetrical about a fiber, F_0 is directly equal to the resultant normal force P_0 on the crack face (Section 5). If the boundary opposite to the crack lies along a fiber, F_0 is equal to $P_0 + M_0/H$, where M_0 is the moment of the crack load about the tip

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and H is the distance from the tip to the opposite boundary (Section 5). Both of these results are obtained with virtually no calculation. A similar result is obtained if the boundary cut by the crack lies along a fiber, but the distance H is replaced by a complicated mean distance (Section 6).

Such simple and explicit results are not available in elastic fracture mechanics. Thus, even if a given real material might be modelled more accurately as a transversely isotropic elastic material, it is possible that the present results may furnish some useful insight. The standing of the present results as approximations to elasticity theory is discussed very briefly in Section 3.

2. EQUATIONS AND BOUNDARY CONDITIONS

We consider plane deformations of a cracked body of ideal fiber-reinforced material. The fibers, which are treated as continuously distributed and inextensible, lie parallel to the x-axis. The crack is perpendicular to the fibers, lying along the positive y-axis with the crack tip at the origin (Fig. 1).

The problem of deformation under prescribed tractions on the external boundary curve C, with no traction on the crack surfaces, can be solved by superposing the solutions of two simpler problems: (i) deformation of the uncracked body by the prescribed tractions on C, and (ii) deformation of the cracked body under no traction on C, but with tractions on the crack sufficient to cancel those given by the solution of the first problem. Methods of solution of the first problem have been discussed elsewhere [3, 6]. Here we consider only the second problem. Thus, we suppose that there is no traction on C, and that on the crack, the stress components σ_{xx} and σ_{xy} take prescribed values:

$$\sigma_{xx}(0, y) = -p(y), \quad \sigma_{xy}(0, y) = -t(y) \quad (y > 0). \tag{2.1}$$

Let $y = h$ be the point at which the crack reaches the outer boundary C, so that h is the length of the crack. For later use, we introduce the following notation:

$$P(y) = \int_y^h p(y')\,dy', \quad M(y) = \int_y^h y'p(y')\,dy'. \tag{2.2}$$

Since the fibers are inextensible, the displacement u in the x-direction is constant along each fiber. We use the notation

$$u(y) = \begin{cases} u_+(y) & (x > 0, y \geq 0), \\ u_-(y) & (x < 0, y \geq 0), \\ u_0(y) & (y \leq 0). \end{cases} \tag{2.3}$$

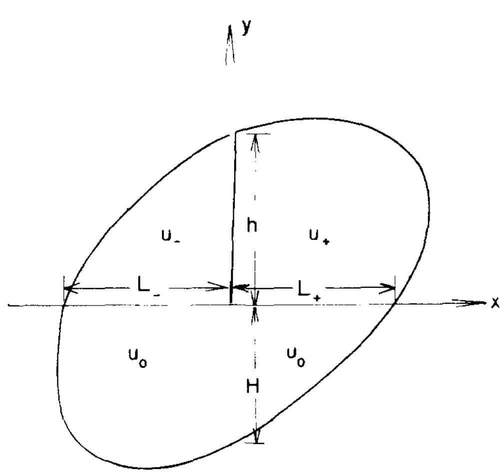

Fig. 1.

The displacement v in the y-direction depends only on x. This follows from the assumption of bulk incompressibility, given fiber inextensibility and the restriction to plane deformation. This idealization is unreasonable for many real composites. The result that $v = v(x)$ would follow more directly if the material contained a second family of inextensible fibers parallel to the y-axis.

Since we consider pure traction problems, the displacement and rotation can be prescribed at one point. We take the displacement at the crack tip to be zero, and in lieu of a rotation condition we take $u'(y)$ to be zero just ahead of the crack tip:

$$u(0) = v(0) = u_0'(0) = 0. \tag{2.4}$$

We suppose that the shearing stress response of the material is elastic, so that

$$\sigma_{xy} = G[u'(y) + v'(x)]. \tag{2.5}$$

The two stress components σ_{xx} and σ_{yy} are reactions to the constraints of fiber inextensibility and bulk incompressibility, and they are determined by using the equilibrium equations and boundary conditions. With (2.5), the equilibrium equations are

$$\sigma_{xx,x} = -\sigma_{xy,y} = -Gu''(y) \tag{2.6}$$

and

$$\sigma_{yy,y} = -\sigma_{yx,x} = -Gv''(x). \tag{2.7}$$

3. ENERGY RELEASE RATE

We wish to calculate the rate at which energy is dissipated in an extension of the crack. In an extension normal to the fibers, the energy release rate is [8]

$$\mathcal{G} = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^{\epsilon} \sigma_{xx}(0, y-\epsilon)\,[u_+(y) - u_-(y)]\,\mathrm{d}y. \tag{3.1}$$

There is no contribution from a discontinuity in v across the crack, because with $v = v(x)$, v is equal to $v(0) = 0$ over the entire length of both faces of the crack. The tangential tractions $t(y)$ on the crack do no work in any admissible deformation, and their sole effect is to produce stress concentration layers along the faces of the crack. These layers have zero thickness and in them the stress σ_{yy} is infinitely large, within the present theory. In a more realistic theory allowing for bulk compressibility, these high stresses would produce a shearing mode of crack deformation, and the energy release rate would include an additional term due to the discontinuity in v. For a compressible material, (3.1) is the contribution to the energy release rate from the opening mode alone.

Within the present theory, the shearing stress is generally discontinuous across the fiber $y = 0$ that passes through the crack tip, and this fiber is a stress concentration layer carrying a finite force. The tensile stress then has the form

$$\sigma_{xx} = F(x)\delta(y) + \sigma^*_{xx}(x, y), \tag{3.2}$$

where F is the force in the singular fiber, $\delta(y)$ is the Dirac delta, and σ^*_{xx} is finite. The relation between F and the discontinuity in $u'(y)$ is found by using (3.2) and (2.3) in the equilibrium eqn (2.6):

$$F'(x) = -G[u_\pm'(0) - u_0'(0)]. \tag{3.3}$$

Let $x = L_+$ and $x = -L_-$ be the points at which the singular fiber $y = 0$ reaches the outer boundary C (Fig. 1). With $F = 0$ at these points, from (3.3) we find that the force $F_0 = F(0)$ at the crack tip is

$$F_0 = GL_+u_+'(0) = -GL_-u_-'(0). \tag{3.4}$$

Here we have used the value $u_0'(0) = 0$ prescribed in (2.4).

When (3.2) is used in (3.1), the finite part σ^*_{xx} makes no contribution in the limit, and we obtain

$$\begin{aligned}\mathcal{G} &= \lim_{\epsilon\to 0} F_0[u_+(\epsilon) - u_-(\epsilon)]/2\epsilon \\ &= \frac{1}{2} F_0[u'_+(0) - u'_-(0)].\end{aligned} \tag{3.5}$$

Then with (3.4), we conclude that

$$\mathcal{G} = F_0^2/GL^*, \tag{3.6}$$

where L^* is the harmonic mean width:

$$2/L^* = 1/L_+ + 1/L_-. \tag{3.7}$$

England and Rogers[4] have suggested fracture criteria that in the present case would predict fracture at a critical value of F_0. If, instead, fracture is predicted to occur at a critical value of the energy release rate, then the fracture criterion is based on the value of $F_0(GL^*)^{-1/2}$ rather than F_0 alone.

The crack-tip stress and displacement fields used here in evaluating the energy release rate are very different from those familiar from elasticity theory[7]. Spencer[9] has considered elastic materials with small but finite fiber extensibility and has used singular perturbation methods to show that the singular fiber through the crack tip can be interpreted as a thin region of high tensile stress, with the expected $0(r^{-1/2})$ stress singularity at the crack tip. In the present theory this point singularity is not seen, since it is buried within the stress concentration layer.

The question arises, whether the energy release rate calculated from the slightly-extensible elastic model must necessarily agree with the result (3.6) from the inextensible model. Sanchez[10] has shown that this is indeed the case; the elastic energy release rate approaches the value (3.6) in the limit as the fiber extensibility approaches zero. We do not prove this in the present paper, but merely accept the result as plausible.

4. DISPLACEMENT EQUATIONS OF EQUILIBRIUM

The result (3.6) is analogous to the expression for the energy release rate in terms of the stress intensity factor in elasticity theory[7]. Just as it is necessary to solve the entire boundary value problem in order to determine the stress intensity factor, here it is often necessary to obtain a complete solution before the fiber force F_0 can be determined. However, solving problems in the present theory is relatively simple in comparison to solving elasticity problems for finite bodies. We give some examples in Sections 5 and 6, after some preliminary manipulations in the present section.

For notational convenience we usually restrict attention to bodies whose outer boundary C is cut no more than twice by any line parallel to a coordinate axis, so that the equation of the boundary can be written in two parts as $y = y_+(x)$ and $y = y_-(x)$, with $y_+ \geq y_-$, or equally well as $x = x_+(y)$ and $x = x_-(y)$, with $x_+ \geq x_-$. We use the following special notation (Fig. 1):

$$x_+(0) = L_+,\ x_-(0) = -L_-,\ y_+(0) = h,\ y_-(0) = -H. \tag{4.1}$$

We also use the notation

$$\Delta x(y) = x_+(y) - x_-(y),\ \Delta y(x) = y_+(x) - y_-(x). \tag{4.2}$$

First integrals of the equilibrium equations can be obtained by the method of shearing stress resultants[3, 6]. The y-component of the resultant force on the part of the boundary to the right of any line x = constant is zero for $x \neq 0$, and thus the total shearing stress on this line is zero. This total can be expressed in terms of displacements by integrating (2.5) over the length of the line:

$$v'(x)\Delta y(x) + u(y_+(x)) - u(y_-(x)) = 0. \tag{4.3}$$

The integral of the shearing stress over the length of any fiber is equal to the x-component of the resultant force applied to the part of the body above that fiber. For fibers that end on the crack, we obtain

$$u'_{\pm}(y)x_{\pm}(y) + v(x_{\pm}(y)) = P(y)/G \quad (0 \leq y \leq h). \tag{4.4}$$

Here we have used (2.2) and (2.4). On fibers that do not intersect the crack, the resultant shearing stress is zero. In particular,

$$u'_0(y)\Delta x(y) + v(x_+(y)) - v(x_-(y)) = 0 \quad (y \leq 0). \tag{4.5}$$

If there is a region $y > h$, the functions $u_{\pm}$ satisfy an equation of the form (4.5) in it.

The problem is posed in terms of u and v by (4.3)–(4.5) and the conditions (2.4). When u has been found, F_0 is given by (3.4). In the latter expression we can evaluate u' in terms of v by using (4.4). In this way we obtain four alternative expressions for F_0:

$$F_0 = Gx_{\pm}(0)u'_{\pm}(0) = P_0 - Gv(x_{\pm}(0)). \tag{4.6}$$

Here P_0 is $P(0)$, the total normal force on the crack.

5. SOME SIMPLE EXAMPLES

There are special cases in which the fiber force F_0 can be evaluated with little or no effort. The simplest cases are those in which the body is symmetrical about a line y = constant, with two symmetrically disposed edge cracks or one symmetrical central crack, symmetrically loaded. England and Rogers[4] have worked out the complete stress and displacement fields for some cases of this sort. For our present purpose, we need only observe that by symmetry, v must vanish on the line of symmetry and thus vanish everywhere. It is not difficult to verify that (4.6) remains valid in this class of examples, with P_0 interpreted as half the resultant pressure in the case of a central crack. With $v = 0$ it follows from (4.6) that $F_0 = P_0$, and from (3.6) it then follows that the energy release rate is P_0^2/GL^*. In view of the inordinate difficulty of solving any one problem in this class by means of elasticity theory, it may be useful to know that the elastic energy release rate is given approximately, when the fiber extensibility is small, by the simple expression P_0^2/GL^*.

Among unsymmetrical bodies, the simplest are those whose lower boundary lies along a fiber. With $y_-(x) = -H$ for all x, and with restriction to boundaries of the type described in Section 4, it is then necessarily true that $x_+(y) \geq L_+$ and $x_-(y) \leq -L_-$ for $y < 0$. For boundaries of this kind, it can be shown that $u_0(y) = 0$ for all $y < 0$. It then follows from (2.6) and the zero-traction boundary conditions that $\sigma_{xx} = 0$ throughout the region $-H < y < 0$. The boundary fiber $y = -H$ may be a singular fiber carrying a finite force.

The foregoing information is enough to allow calculation of F_0. We consider the part of the body to the right of the line $x = 0$, and calculate the moments of the forces on this part about the pivot $x = 0$, $y = -H$. The load on the crack surface has moment $M_0 + P_0H$, where $M_0 = M(0)$ is the moment about the crack tip, defined in (2.2). This must be balanced by the moment of the contact forces exerted by the left-hand part of the body on the right-hand part, across the line $x = 0$. The shearing stresses on this line have no moment about the chosen pivot, which lies on the line, and similarly the force in the boundary fiber has no moment. Since $\sigma_{xx} = 0$ for $-H < y < 0$, the total moment is that of the crack-tip fiber force, F_0H. Consequently,

$$F_0 = P_0 + M_0/H. \tag{5.1}$$

6. A MORE COMPLICATED CLASS OF EXAMPLES

We now consider a class of examples in which it is necessary to solve the system (4.3)–(4.5) before F_0 can be found. We consider bodies with a straight upper boundary, $y_+(x) = h$, and a convex lower boundary, $y = y(x)$ (we omit the subscript on $y_-(x)$). For all such bodies we obtain a result of the general form (5.1), but with H replaced by a certain average thickness. This result is given in (6.17).

Part of the body may lie below the line $y = -H$. Since the lower boundary is convex, this part must lie entirely on one side of the line $x = 0$ which intersects the boundary at $y = -H$. We take the positive x-axis to point away from such a part, if there is one, so that $y(x)$ increases monotonically with x for $x > 0$. Then in the right-hand part of the body there is a one-to-one correspondence between the values of x and y that are coupled in the equilibrium eqns (4.3)–(4.5). In determining F_0 it will prove to be unnecessary to find u and v in the region $y < -H$.

Analysis of the system of differential-difference eqns (4.3)–(4.5) is facilitated by introducing the difference quotients Q_+ and Q_- defined by

$$Q_\pm(y) = \begin{cases} [u_\pm^* - u_\pm(y)]/(h-y) & (y \geq 0), \\ [u_\pm^* - u_0(y)]/(h-y) & (-H \leq y \leq 0). \end{cases} \tag{6.1}$$

Here $u_\pm^* = u_\pm(h)$. The quotient is the difference between the boundary values of u at the two ends of a line x = constant, divided by the distance between the two ends. The displacement u is given in terms of the quotient by

$$\begin{aligned} u_\pm(y) &= u_\pm^* + (y-h)Q_\pm(y) \quad (y \geq 0) \\ u_0(y) &= u_+^* + (y-h)Q_+(y) \quad (-H \leq y \leq 0). \end{aligned} \tag{6.2}$$

The boundary conditions on u in (2.4) are, in terms of Q,

$$Q_\pm(0) = u_\pm^*/h \quad \text{and} \quad Q_\pm'(0-) = u_\pm^*/h^2. \tag{6.3}$$

We intend to eliminate v from the system (4.3)–(4.5) by first differentiating (4.4) and (4.5) and then using (4.3). Because of the preliminary differentiation, the resulting system is not fully equivalent to the original system. It is necessary to insure in addition that (4.4) and (4.5) are satisfied at some one point, say $y = 0$. These three conditions are equivalent to the stipulation that all four of the expressions in terms of u and v in (4.6) have the same (unknown) value F_0. When the two conditions involving u in (4.6) are expressed in terms of Q, we obtain

$$Q_\pm'(0+) = u_\pm^*/h^2 - F_0/Ghx_\pm(0). \tag{6.4}$$

In addition, the two conditions on v in (4.6) remain to be satisfied.

We need to use (4.3) only for values of x equal to $x_\pm(y)$ with $y > -H$. In this domain of y, x_+ is positive and x_- is negative. Consequently, (4.3) can be written as

$$v'(x_\pm(y)) = -Q_\pm(y) \qquad (y \geq -H). \tag{6.5}$$

By differentiating (4.4) and (4.5) with respect to y and expressing u in terms of Q in the resulting expressions, we obtain

$$[(y-h)x_\pm Q_\pm' + x_\pm Q_\pm]' + x_\pm' v'(x_\pm) = -p(y)/G \qquad (y \geq 0) \tag{6.6}$$

and

$$[(y-h)\Delta x Q_+' + \Delta x Q_+]' + x_+' v'(x_+) - x_-' v'(x_-) = 0 \quad (-H \leq y \leq 0). \tag{6.7}$$

We use (6.5) to eliminate v from (6.6) and (6.7). The resulting equations have an integrating factor $y - h$, and can be put into the forms

$$[(y-h)^2 x_\pm Q_\pm']' = (h-y)p(y)/G \quad (y \geq 0) \tag{6.8}$$

and

$$[(y-h)^2 \Delta x Q_+']' = x_-'(Q_+ - Q_-)(y-h) = -x_-'(u_+^* - u_-^*) \quad (-H \leq y \leq 0). \tag{6.9}$$

The last member of (6.9) is obtained from the second by using (6.1).

We now integrate (6.5), (6.8) and (6.9). From (6.5) we ultimately need only the value $v(L_+)$. To obtain an expression for this value, we multiply (6.5) by x_+' and then integrate from $y = -H$, where $x_+ = 0$, to $y = 0$, where $x_+ = L_+$. We recall that $v(0) = 0$. Then, with an integration by parts, we obtain

$$v(L_+) = -L_+Q_+(0) + \int_{-H}^{0} x_+(y)Q_+'(y)\,dy. \tag{6.10}$$

In integrating (6.8) we use the boundary condition that $Q_\pm'(h)$ is finite, and obtain

$$G(y-h)^2 x_\pm Q_\pm' = M(y) - hP(y) \qquad (y \ge 0) \tag{6.11}$$

where M and P are defined in (2.2). The boundary condition (6.3b) is used in integrating (6.9). We obtain

$$(y-h)^2 \Delta x Q_+' = u_\mp^* \Delta x(0) - (u_\mp^* - u_\pm^*)\,[x_-(y) - x_-(0)]\,(-H \le y \le 0). \tag{6.12}$$

This yields the function Q_+' to be used in (6.10). The final integrations of (6.11) and (6.12) are unnecessary in finding F_0.

It remains to determine the parameters $u_\pm^*$ and F_0. By using (6.11) in (6.4), we obtain expressions for $u_\pm^*$ in terms of F_0:

$$Gx_\pm(0)u_\pm^* = M_0 - hP_0 + hF_0. \tag{6.13}$$

We use this to eliminate $u_\pm^*$ from (6.12):

$$G(y-h)^2\Delta x Q_+' = -(M_0 - hP_0 + hF_0)\,(2/L^*)x_-(y) \quad (-H \le y \le 0). \tag{6.14}$$

Here L^* is defined in (3.7). We next use (6.3a), (6.13) and (6.14) in (6.10), and obtain

$$Gv(L_+) = (2I/L^* - 1/h)\,(M_0 - hP_0 + hF_0). \tag{6.15}$$

Here I is the integral defined by

$$I = \int_{-H}^{0} [-x_-x_+/\Delta x(y-h)^2]\,dy. \tag{6.16}$$

Finally, we use (6.15) in (4.6b) and solve for F_0. We obtain

$$F_0 = P_0 + M_0/D, \tag{6.17}$$

where D is a length defined by

$$D = 2h^2 I/(L^* - 2hI). \tag{6.18}$$

We can apply both (5.1) and (6.17) to rectangular bodies, and the two formulas agree if $D = H$ for such bodies. It is straightforward to verify that (6.16) and (6.18) do yield the result $D = H$ in such cases.

Some further understanding of the nature of the length D can be obtained by considering bodies symmetrical about the crack, for which $x_- = -x_+$ and $L_\pm = L$, say. For these cases, an integration by parts in (6.16) yields

$$I = L/2h - L/2H^*, \tag{6.19}$$

where H^* is the harmonic mean width defined by

$$\frac{1}{H^*} = \frac{1}{L}\int_0^L \frac{dx}{h - y(x)}. \tag{6.20}$$

Then from (6.18), D is found to be $H^* - h$. In both this and the preceding special case, D can be interpreted as an average depth of the uncracked part of the body.

7. DISCUSSION

Since the direction of extension of a crack need not be parallel to its original direction, the parallel extensions considered in the present paper represent only the simplest possibility. In the case of a crack parallel to the fibers, it appears plausible that further extension will remain parallel to them. However, we have chosen to phrase our discussion in terms of a crack perpendicular to the fibers. In this case it is immediately evident that the crack might become L-shaped, with further extension by delamination rather than by fiber-breaking. This possibility would appear to be particularly likely when the shearing stresses near the crack tip are large.

In this connection we note that for the particular classes of body shapes discussed in Sections 5 and 6, it is possible to calculate the crack-tip shearing stresses explicitly. As an example, we consider the especially simple case of a symmetrical body, symmetrically loaded (Section 5). In this case v is zero and thus $\sigma_{xy} = Gu'(0)$ near the tip. Since $u_0'(0) = 0$ and $u_\pm'(0) = \pm F_0/GL_\pm$ (from (3.4)), there is no shearing stress just ahead of the tip but there are stresses $\sigma_{xy} = \pm F_0/L_\pm$ just behind it. If a fracture criterion in terms of shearing stress be applicable, the crack will extend in the positive x-direction if L_+ is smaller than L_-.

However, there is a reason to doubt that the crack-tip shearing stresses determine the direction of crack advance. If the idealizations used in the present theory are taken quite literally, there can be no in-plane shearing mode of crack extension, and the shearing stresses make no direct contribution to the energy release rate. The general expression (3.6) for the energy release rate remains valid for an L-shaped crack, with suitable reinterpretation of the quantities in it. For an incipient extension in the positive x-direction, F_0 should be replaced by F_+, the force in the singular normal line along the right-hand face of the crack. This force can be expressed in terms of the tangential load $t(y)$ on the crack free and the shearing stress σ_{xy} just inside the material by

$$F_+ = \int_0^h (\sigma_{xy} + t)\, \mathrm{d}y. \tag{7.1}$$

Thus, if the direction of crack advance depends on a criterion in terms of the energy release rate, it is not sufficient to consider only the crack-tip shearing stresses.

Within the present theory, it should be possible to calculate the energy release rate for an arbitrary direction of crack extension. We might then be able to predict the direction of crack advance, by assuming that it is the direction of maximum energy release rate. We intend to carry out further work along these lines.

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